Suggested Solutions for HW9

8.3 Q5

5. If $x \geq 0$ and $n \in \mathbb{N}$, show that

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \frac{(-x)^n}{1+x}.$$

Use this to show that

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

and that

$$\left|\ln(x+1) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \leq \frac{x^{n+1}}{n+1}.$$

Sol : Note that $\frac{1-(-x)^n}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1}$

which implies $\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \frac{(-x)^n}{1+x}$

Also by the observation $\int_0^x \frac{1}{1+t}\,dt = \ln(1+x)$

Then $\ln(1+x) = \int_0^x 1 - t + t^2 - t^3 + \cdots + (-t)^{n-1} + \frac{(-t)^n}{1+t}\,dt$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

And $\left|\ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| = \left|\int_0^x \frac{(-t)^n}{1+t}\,dt\right|$

But $\left|\int_0^x \frac{(-t)^n}{1+t}\,dt\right| \leq \int_0^x \left|\frac{(-t)^n}{1+t}\right|\,dt$

$$\leq \int_0^x t^n\,dt$$

$$= \frac{x^{n+1}}{n+1}$$

8.4 Q7

7. Show that the functions c, s in the preceding exercise have derivatives of all orders, and that they satisfy the identity $(c(x))^2 - (s(x))^2 = 1$ for all $x \in \mathbb{R}$. Moreover, they are the unique functions satisfying (j) and (jj). (The functions c, s are called the **hyperbolic cosine** and **hyperbolic sine functions**, respectively.)

Sol: Recall the functions c, s defined in Q6 satisfy

① $c(0)=1$, $c'(0)=0$, $s(0)=0$, $s'(0)=1$
② $c'(x)=s(x)$, $s'(x)=c(x)$ for all $x \in \mathbb{R}$
③ $c''(x)=c(x)$, $s''(x)=s(x)$ for all $x \in \mathbb{R}$

By ②, it's clear that c, s have derivatives of all orders

Let $f(x) = (c(x))^2 - (s(x))^2$

Then $f'(x) = 2c(x)c'(x) - 2s(x)s'(x)$

$= 2c(x)s(x) - 2s(x)c(x)$

$= 0$ for all $x \in \mathbb{R}$

Therefore, $f(x) \equiv f(0) = 1$

For the uniqueness part, let c_1, c_2 be two functions satisfy ①, ③ and let $\varphi = c_1 - c_2$

Then $\varphi''(x) = \varphi(x)$ $\forall x \in \mathbb{R}$ and $\varphi(0) = \varphi^{(k)}(0) = 0$ $\forall k \in \mathbb{N}$

Now pick any $x \in \mathbb{R} \setminus \{0\}$, let $I_x = [0, x]$ (or $[x, 0]$ if $x<0$)

By Taylor's Thm, for each $n \in \mathbb{N}$, there exists $x_n \in I_x$ s.t.

$$\varphi(x) = \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(n)}(x_n)}{n!} x^n$$

$$= \frac{\varphi^{(n)}(x_n)}{n!} x^n$$

Since φ and φ' are continuous on I_x, then there exists $K>0$

s.t. $|\varphi(t)| < K$ and $|\varphi'(t)| < K$ for all $t \in I_x$

It follows that $|\varphi^{(n)}(t)| < K$ for all $n \in \mathbb{N}$, $t \in I_x$

Moreover, $\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$

which implies $\varphi(x) = 0$ for all $x \neq 0$

Together with the fact $\varphi(0) = 0$, we have $\varphi(x) \equiv 0$, $\forall x \in \mathbb{R}$

We infer that $C_1(x) = C_2(x)$ $\forall x \in \mathbb{R}$

The same argument also applies to the uniqueness of $S(x)$

9.1 Q1

1. Show that if a convergent series contains only a finite number of negative terms, then it is absolutely convergent.

Sol: Suppose the convergent series $\sum a_n$ contains only a finite number of negative terms

Write $A = \sum a_n$ and B denote the sum of the negative terms

where A and B are both real numbers

Then $\sum |a_n| = A - 2B < \infty$

9.1 Q6

6. Find an explicit expression for the nth partial sum of $\sum_{n=2}^{\infty} \ln\left(1 - 1/n^2\right)$ to show that this series converges to $-\ln 2$. Is this convergence absolute?

Sol: Note that $\ln\left(1-\frac{1}{n^2}\right) = \ln(n^2-1) - 2\ln n$

$= \ln(n+1) + \ln(n-1) - 2\ln n \quad \forall n \geq 2$

Now pick any integer $N \geq 2$

$$\sum_{n=2}^{N} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{N} \left[\ln(n+1) + \ln(n-1) - 2\ln n\right]$$

$$= \sum_{n=1}^{N-1} \ln n + \sum_{n=3}^{N+1} \ln n - 2\sum_{n=2}^{N} \ln n$$

$$= -\ln 2 + \ln(N+1) - \ln N$$

$$= -\ln 2 + \ln\left(1+\frac{1}{N}\right)$$

Hence, $$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \lim_{N\to\infty} \sum_{n=2}^{N} \ln\left(1-\frac{1}{n^2}\right)$$

$$= -\ln 2 + \lim_{N\to\infty} \ln\left(1+\frac{1}{N}\right)$$

$$= -\ln 2$$

Since all the terms $\ln\left(1-\frac{1}{n^2}\right)$ are negative, the series is also absolutely convergent and $\sum\left|\ln\left(1-\frac{1}{n^2}\right)\right| = \ln 2$

9.1 Q8

8. Give an example of a convergent series $\sum a_n$ such that $\sum a_n^2$ is not convergent. (Compare this with Exercise 3.7.11.)

Sol : Let $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$, then $\sum a_n^2 = \sum \frac{1}{n}$ is divergent

In the following, we wish to show $\sum a_n$ is convergent by the Comparison test

Note that
$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{(n+1)n}}$$
$$= \frac{1}{\sqrt{(n+1)n}\,(\sqrt{n+1} + \sqrt{n})}$$
$$\leq \frac{1}{\sqrt{n^2}\,(2\sqrt{n})}$$
$$= \frac{1}{2} n^{-\frac{3}{2}}$$

Let
$$b_n = \begin{cases} a_n + a_{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$
$$c_n = \begin{cases} \frac{1}{2} n^{-\frac{3}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{is } n \text{ is even} \end{cases}$$

Then $b_n \leq c_n \leq \frac{1}{2} n^{-\frac{3}{2}}$, $\forall n \geq 1$, and $\sum a_n = \sum b_n$

Since $\sum_{n=1}^{\infty} \frac{1}{2} n^{-\frac{3}{2}}$ is convergent and by the comparison test, $\sum b_n$ and $\sum c_n$ are convergent, which also implies the convergence of $\sum a_n$

9.1 Q12

12. Let $a > 0$. Show that the series $\sum(1+a^n)^{-1}$ is divergent if $0 < a \leq 1$ and is convergent if $a > 1$.

Sol: ① If $0 < a \leq 1$, then $0 < a^n \leq 1$ for all $n \geq 1$

Hence $\frac{1}{2} \leq (1+a^n)^{-1} < 1$, $\forall n \geq 2$

And $\sum(1+a^n)^{-1} \geq \sum \frac{1}{2} = \infty$ is divergent

② if $a > 1$, then $0 < (1+a^n)^{-1} < a^{-n}$, $\forall n \geq 1$

Since $\sum a^{-n} = \frac{1}{1-a^{-1}}$ is convergent if $a > 1$, and by the comparison Test, $\sum(1+a^n)^{-1}$ is convergent